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Polynomials with only real zeros

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ABSTRACT

The conditions for a sequence of polynomials to have only real zeros based on the method of interlacing zero was investigated by Lily L. Liu and Yi Wang. In this paper we will find new conditions for a combination of real polynomials, continues to have only real zeros based on the method of interlacing zero.

Keywords: Polynomials, real zeros, alternate, interlace, l-alternates.

1. Introduction

Polynomials with only real zeros arise often in combinatorics and other branches of mathematics.We refer the reader to the survey papers by Wang and Yeh (2005) and Liu and Wang (2007) for various result for polynomials to have only real zeros. Following Wagner (1992), let $RZ = \{P(x) \in \mathbb{R}[x]; P(x)$ has only real zeros$\}$ .Suppose that $f, g \in RZ$. Let $\{r_i\}$ and $\{s_j\}$ be all zeros of f and g in non-increasing order respectively.We say that g *alternates* f if $\deg f = \deg g = n$ and

$$s_n \leq r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{1}$$

We say that g interlaces f if $\deg f = \deg g + 1 = n$ and

$$r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{2}$$

The notation $g \preceq f$ denote either g alternates f or g interlaces f. If no equality sign occurs in (1)(respectively (2)), then we say that g strictly alternates f (respectively g strictly interlaces f).Let $g \prec f$ denote either g strictly alternates f or g strictly interlaces f. For notational convenience, let $a \preceq bx+c$ for any real constants a, b, c and $f \preceq 0, 0 \preceq f$ for any real polynomial f with only real zeros.

We say that g l-alternates f if $\deg f = \deg g + 2 = n + 1$ and

$$r_{n+1} \leq r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{3}$$

If no equality sign occurs in (3), then we say that g strictly l-alternates f.

Let f and g be two real polynomials whose leading coefficients have the same sign. Suppose that f and g have only real zeros and that g alternates f or g interlaces f.Wang and Yeh (2005) found that the polynomial $(bx + a)f(x) + (dx + c)g(x)$ has only real zeros where $ad \geq bc$. Liu and Wang (2007) proved this following result.

Theorem 1.1. *Let F, f, g be three real polynomials satisfying the following conditions;*
(a) $F(x) = a(x)f(x) + b(x)g(x)$ where $a(x), b(x)$ are two real polynomials, such that $\deg F = \deg f$ *or* $\deg f + 1$,
(b) $f, g \in RZ$ and $g \preceq f$,
(c) F and g have leading coefficients of the same sign,
(d) $\forall r \in \mathbb{R}, f(r) = 0 \Rightarrow b(r) \leq 0$.
Then $F \in RZ$ and $f \preceq F$. In particular, if $g \prec f$ and $b(r) < 0$ whenever $f(r) = 0$, then $f \prec F$.

They also gave a short and simple proof for this following result of Haglund (2000).

Theorem 1.2. *Let f and g be two real polynomials with both positive leading coefficients α and β respectively. Suppose that the following conditions are satisfied;*
(a) f , $g \in RZ$ and g interlaces f,
(b) $F(x) = (ax + b)f(x) + x(x + d)g(x)$ where $a, b, c, d \in R$ with $d \geq 0, d \geq b/a$ and either $a > 0$ or $a < -\beta/\alpha$,
(c) All zeros of f are non-positive if $a > 0$ and nonnegative if $a < -\beta/\alpha$.
Then $F \in RZ$. In addition, if each zeros r of f satisfies and $-d \leq r \leq 0$, then f interlaces F.

Haglund (2000)used Theorem 1.2 to prove facts about rook polynomials in graph theory. Srimud et al. (2011) proved a generalization of Theorem 1.2, viz.,

Theorem 1.3. *Let f and g be two real polynomials with both positive or negative leading coefficients α and β respectively. Suppose that the following conditions are satisfied;*

(a) f , $g \in RZ$ and g interlaces f,
(b) $F(x) = (ax+b)f(x)+x(cx+d)g(x)$ where $a, b, c, d \in \mathbb{R}$ with $a \neq 0, d \geq bc/a$.
(c) if $a > 0$, then all zero of f is nonpositive,
(d) if $a < 0$, then all zero of f is nonnegative.

Then $F \in RZ$. In addition,
if $c > 0$ and $-d/c \leq r \leq 0$ each zero r of f, then f interlaces F.
if $c < 0$ and $r \leq -d/c \leq 0$ each zero r of f, then f interlaces F.
if $c = 0$ and $r \leq 0$ for each zero r of f, then f interlaces F.

It is natural to ask for other conditions ensuring positivity of coefficients. Here, we derive analogous conditions for a family of such polynomials.

2. Main Results

Let

$$\operatorname{sgn}(x) = \begin{cases} +1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases}$$

Let $f(x)$ be real function and write $\text{sgn}f(+\infty) = +1$ and -1, respectively, if $\text{sgn}f(x) = +1$ and -1 for sufficiently large x. Similar definition is given for $\text{sgn}f(-\infty)$

Our main result is

Theorem 2.1. *Let F, f, g be three real polynomials satisfying the following conditions;*

(a) $F(x) = a(x)f(x) + b(x)g(x)$ where $a(x), b(x)$ are two real polynomials, such that $b(x) \not\equiv 0$ and $\deg F = \deg f + 2$,
(b) $f, g \in RZ$ and g alternates f,
(c) F and g have leading coefficients of the different sign,
(d) $\forall r \in \mathbb{R}, f(r) = 0 \Rightarrow b(r) \geq 0$,
(e) if f and g have leading coefficient of opposite sign, then $a(s) < 0$ where s is a root of least value of g. If f and g have leading coefficients of the same sign, then $a(s) > 0$ where s is a root of least value of g.

Then $F \in RZ$ and f l-alternates F. In addition, if g strictly alternates f and $b(r) > 0$ whenever $f(r) = 0$, then f strictly l-alternates F.

Proof. Let F, f, g be three real polynomials with $\deg f = n$, so $\deg F = n + 2$. Since g alternates f, we have $\deg g = \deg f = n$. Since $f, g \in RZ$, there exists $r_i, s_j \in \mathbb{R}$ such that $f(x) = k(x - r_1)(x - r_2) \cdots (x - r_n)$ and $g(x) = l(x - s_1)(x - s_2) \cdots (x - s_n)$, where $1 \leq i, j \leq n$.
First, consider the case where g strictly alternate f and $b(r) > 0$ whenever $f(r) = 0$. Suppose that $s_n < r_n < \cdots < s_2 < r_2 < s_1 < r_1$. Let $i \in \{1, 2, \ldots, n\}$, and let the leading coefficients of g be negative and leading coefficients of F be positive. Since $g(r_i) = l(r_i - s_1)(r_i - s_2) \cdots (r_i - s_i)(r_i - s_{i+1}) \cdots (r_i - s_n)$, we have sgn $g(r_i) = (-1)^i$. Thus, $F(r_i) = a(r_i)f(r_i) + b(r_i)g(r_i)$, and so sgn $F(r_i) = (-1)^i$. Since the leading coefficient of F is positive, then sgn $F(+\infty) = +1$ and sgn $F(-\infty) = (-1)^{\deg F} = (-1)^{n+2}$. By the Intermediate Value Theorem, there exist $u_{i+1} \in (r_{i+1}, r_i)$ such that $F(u_{i+1}) = 0$ for all $i, 1 \leq i \leq n-1$. Then $r_n < u_n < r_{n-1} < \cdots < r_2 < u_2 < r_1$. Since $\text{sgn}F(r_1) = -1$ and $\text{sgn}F(+\infty) = 1$, then there exists u_1 such that $r_1 < u_1$.
Consider

$$\begin{aligned} F(s_n) &= a(s_n)f(s_n) + b(s_n)g(s_n), \\ &= a(s_n)f(s_n), \\ &= a(s_n)k(s_n - r_1)(s_n - r_2) \cdots (s_n - r_n). \end{aligned}$$

If $k < 0$ and $a(s_n) > 0$, then $\text{sgn}F(s_n) = (-1)^{n+1}$.
If $k > 0$ and $a(s_n) < 0$, then $\text{sgn}F(s_n) = (-1)^{n+1}$.
In both cases, we have $\text{sgn}F(s_n) = (-1)^{n+1}$, and so $F(x)$ has two additional zeros $u_{n+1}, u_{n+2} \in \mathbb{R}$ in intervals (s_n, r_n) and $(-\infty, s_n)$, i.e., $u_{n+2} < u_{n+1} < r_n < u_n < r_{n-1} < \cdots < u_2 < r_1 < u_1$, showing that f strictly l-alternate F.
For the general case, let $b_j(x) = b(x) + 1/j$ and $F_j(x) = a(x)f(x) + b_j(x)g(x)$. Let j be sufficiently large. Then $b_j(r_i) = b(r_i) + 1/j > b(r_i) \geq 0$ for all $i, 1 \leq i \leq n$. Thus, $F_j \in RZ$ and f strictly l-alternates F_j.
Since

$$\begin{aligned}
F_j(x) &= a(x)f(x) + b_j(x)g(x), \\
&= a(x)f(x) + (b(x) + 1/j)g(x), \\
&= a(x)f(x) + b(x)g(x) + g(x)/j, \\
&= F(x) + g(x)/j,
\end{aligned}$$

we have $\deg F_j = \deg F$. Suppose that $u_i^{(j)}$ be root of F_j for $1 \leq i \leq n+2$. Since f strictly l-alternates F_j, we get

$$u_{n+2}^{(j)} < u_{n+1}^{(j)} < r_n < u_n^{(j)} < \cdots < u_2^{(j)} < r_2 < u_1^{(j)}$$

Let $F(x) = a_{n+2}x^{n+2} + a_{n+1}x^{n+1} + \cdots + a_1x + a_0$. and $F_j(x) = a_{n+2}^{(j)}x^{n+2} + \cdots + a_1^{(j)}x + a_0^{(j)}$.
We have

$$\begin{aligned}
\lim_{j\to\infty} F_j(x) &= \lim_{j\to\infty} \Big(F(x) + \frac{g(x)}{j}\Big), \\
&= \lim_{j\to\infty} (a_{n+2}^{(j)}x^{n+2} + \cdots + a_1^{(j)}x + a_0^{(j)}), \\
&= \lim_{j\to\infty} \Big(a_{n+2}x^{n+2} + (a_n + \frac{l_n}{j})x^n + \cdots + (a_0 + \frac{l_0}{j})\Big),
\end{aligned}$$

where $g(x) = l_nx^n + l_{n-1}x^{n-1} + \cdots + l_0$. Thus, $\lim_{j\to\infty} a_i^{(j)} = \lim_{j\to\infty} a_i$.
Note that the zeros of a polynomial are continuous function of coefficients of the polynomial, i.e.,

$$\begin{aligned}
\lim_{j\to\infty} r_i^{(j)} &= h\Big(\lim_{j\to\infty} a_0^{(j)}, \cdots, \lim_{j\to\infty} a_{n+2}^{(j)}\Big), \\
&= h(a_0, a_1, \cdots, a_{n+2}), \\
&= r_i \qquad \text{, when } r_i^{(j)} \text{are the roots of } F_j(x)
\end{aligned}$$

Thus, $u_{n+2} \leq u_{n+1} \leq r_n \leq u_n \leq r_{n-1} \leq \cdots \leq u_2 \leq r_1 \leq u_1$. Hence f l-alternates F.

If the leading coefficients of g be positive and leading coefficients of F be negative, , we have $\text{sgn}g(r_i) = (-1)^{i-1}$ for all $1 \leq i \leq n$, and so $\text{sgn}F(r_i) = (-1)^{i-1}$. Since the leading coefficient of F is negative and $\deg F = n+2$, we have $\text{sgn}F(+\infty) = -1$ and $\text{sgn}F(-\infty) = (-1)^{n+3} = (-1)^{n+1}$.
By the Intermediate Value Theorem, there exist $u_{i+1} \in (r_{i+1}, r_i)$ such that $F(u_{i+1}) = 0$ for all $i, 1 \leq i \leq n-1$. Thus, $r_n < u_n < r_{n-1} < \cdots < r_2 < u_2 < r_1$.
Since $\text{sgn}F(r_1) = 1$ and $\text{sgn}F(+\infty) = -1$, then there exists u_1 such that $r_1 < u_1$.
Consider

$$\begin{aligned} F(s_n) &= a(s_n)f(s_n) + b(s_n)g(s_n), \\ &= a(s_n)f(s_n), \\ &= a(s_n)k(s_n - r_1)(s_n - r_2)\cdots(s_n - r_n). \end{aligned}$$

If $k < 0$ and $a(s_n) < 0$,then $\text{sgn}F(s_n) = (-1)^n$.
If $k > 0$ and $a(s_n) > 0$,then $\text{sgn}F(s_n) = (-1)^n$.
In both cases, we have $\text{sgn}F(s_n) = (-1)^n$, and so $F(x)$ has two additional zeros $u_{n+1}, u_{n+2} \in \mathbb{R}$ in intervals (s_n, r_n) and $(-\infty, s_n)$, i.e., $u_{n+2} < u_{n+1} < r_n < u_n < r_{n-1} < \cdots < u_2 < r_1 < u_1$, showing that f strictly l-alternate F.

The case where $b(r) \geq 0$whenever $f(r) = 0$ can be proved in the same manner. We end our discussion with an example an a corollary. □

Example 2.1. *Let $a(x) = x(x+2), f(x) = 2x(x-2), b(x) = x^2$ and $g(x) = -(x-1)(x+1)$. Here,*

$$\begin{aligned} F(x) &= a(x)f(x) + b(x)g(x) \\ &= x(x+2)2x(x-2) + x^2(-(x-1)(x+1)) \\ &= 2x^2(x^2-4) - x^2(x^2-1) \\ &= x^2(2x^2 - 8 - x^2 + 1) \\ &= x^2(x^2-7). \end{aligned}$$

The roots of F are $0, 0, \sqrt{7}, -\sqrt{7}$ and satisfy

$$-\sqrt{7} < 0 \leq 0 \leq 0 < 2 < \sqrt{7}$$

, i.e., $F \in RZ$ and f l-alternates F.

Corollary 2.1. *Let F, f, g be three real polynomials satisfying the following conditions;*

(a) $F(x) = a(x)f(x) + b(x)g(x)$ *where* $a(x), b(x)$ *are two real polynomials, such that* $b(x) \not\equiv 0$ *and* $\deg F = \deg f + 2$,
(b) $f, g \in RZ$ *and* g *alternates* f,
(c) F *and* g *have leading coefficients of the different sign,*
(d) $\forall r \in \mathbb{R}, f(r) = 0 \Rightarrow b(r) \geq 0$,
(e) if f *has positive leading coefficient, then* $a(s) < 0$ *where* s *is a root of least value of* g *and if* f *and* g *has negative leading coefficients, then* $a(s) > 0$ *where* s *is a root of least value of* g.

Then $F \in RZ$ *and* f *l-alternates* F. *In addition, if* g *strictly alternates* f *and* $b(r) < 0$ *whenever* $f(r) = 0$, *then* f *strictly l-alternates* F.

Proof. The proof is similar to that of Theorem 2.1 □

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